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FAITHFUL LINEAR REPRESENTATIONS OF THE BRAID GROUPS

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The braid group on n strings B_n can be defined as the group generated by $n-1$ generators $\sigma_1, ..., \sigma_{n-1}$ with defining relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \tag{0.1}$$

if $|i-j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{0.2}$$

for $i = 1, ..., n-2$. This group was introduced by Emil Artin in 1926. It has various interpretations, specifically, as the group of geometric braids in $\mathbf{R}^3$, as the mapping class group of an n-punctured disc, as the fundamental group of the configuration space of n points on the plane, etc. The algebraic properties of B_n have been studied by many authors. To mention a few, note the solution of the conjugacy problem in B_n given by F. Garside, the papers of N. Ivanov and J. McCarthy who proved that the mapping class groups and in particular B_n satisfy the "Tits alternative", and the work of P. Dehornoy establishing the existence of a left-invariant total order on B_n (see [Ga], [Iv2], [Ka]).

One of the most intriguing problems in the theory of braids is the question of whether B_n is linear, i.e., whether it admits a faithful representation into a group of matrices over a commutative ring. This question has its origins in a number of interrelated facts and first of all in the discovery by W. Burau [Bu] of an n-dimensional linear representation of B_n which for a long time had been considered as a candidate for a faithful representation. However, as it was established by J. Moody in 1991 this representation is not faithful for $n \geq 9$. Later it was shown to be unfaithful for $n \geq 5$. Thus, the question of the linearity of B_n remained open.

In 1999/2000 there appeared a series of papers of D. Krammer and S. Bigelow who proved that B_n is linear for all n. First there appeared a paper of Krammer

[Kr1] in which he constructed a homomorphism from B_n to $GL(n(n-1)/2, R)$ where $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ is the ring of Laurent polynomials on two variables. He proved that this homomorphism is injective for $n = 4$ and conjectured that the same is true for all n. Soon after that, Bigelow [Bi2] gave a beautiful topological proof of this conjecture. Another proof based on different ideas was obtained by Krammer [Kr2] independently.

The ring $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ can be embedded in the field of real numbers by assigning to q, t any algebraically independent non-zero real values. Therefore B_n embeds in $GL(n(n-1)/2, \mathbf{R})$. As an application, note that the linearity of B_4 implies the linearity of the group $\mathrm{Aut}(F_2)$, where F_n is a free group of rank n (see [DFG]). It is known that $\mathrm{Aut}(F_n)$ is not linear for $n \geq 3$, see [FP].

The representation of B_n considered by Krammer and Bigelow is one of a family of representations introduced earlier by R. Lawrence [La]. Her work was inspired by a study of the Jones polynomial of links and was concerned with representations of Hecke algebras arising from the actions of braids on homology of configuration spaces.

The same representation of B_n arises from a study of the so-called Birman-Murakami-Wenzl algebra C_n. This algebra is a quotient of the group ring $\mathbf{C}[B_n]$ by certain relations inspired by the theory of link polynomials. The irreducible finite dimensional representations of C_n were described by H. Wenzl in terms of Young diagrams. These representations yield irreducible finite dimensional representations of B_n. One of them was shown by M. Zinno [Zi] and independently by V. Jones to be equivalent to the Krammer representation which is henceforth irreducible.

The aim of this paper is to present these results. In Sect. 1 we consider the Burau representation and explain why it is not faithful. In Sect. 2 we outline Bigelow's approach following [Bi2]. In Sect. 3 we discuss the work of Krammer [Kr2]. Finally in Sect. 4 we discuss the Birman-Murakami-Wenzl algebras and the work of Zinno.

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1. THE BURAU REPRESENTATION OF THE BRAID GROUP

1.1. Mapping class groups. It will be convenient for us to view the braid group as the mapping class group of a punctured disc. We recall here the definition and a few simple properties of the mapping class groups.

Let Σ be a connected oriented surface. By a *self-homeomorphism* of Σ we mean an orientation preserving homeomorphism $\Sigma \to \Sigma$ which fixes $\partial\Sigma$ pointwise. Two such

homeomorphisms are *isotopic* if they can be included in a continuous one-parameter family of self-homeomorphisms of Σ. The mapping class group $\mathrm{Homeo}(\Sigma)$ of Σ is the group of isotopy classes of self-homeomorphisms of Σ with the group operation determined by composition.

Each self-homeomorphism of Σ induces an automorphism of the abelian group $H = H_1(\Sigma; \mathbf{Z})$. This is a "homological" representation $\mathrm{Homeo}(\Sigma) \to \mathrm{Aut}(H)$. The action of homeomorphisms preserves the skew-symmetric bilinear form $H \times H \to \mathbf{Z}$ determined by the algebraic intersection number. The value $[\alpha] \cdot [\beta] \in \mathbf{Z}$ of this form on the homology classes $[\alpha], [\beta] \in H$ represented by oriented loops α, β on Σ is computed as follows. Assume that α and β lie in a generic position so that they meet each other transversely in a finite set of points which are not self-crossings of α or β. Then

$$[\alpha] \cdot [\beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p$$

where $\varepsilon_p = +1$ if the tangent vectors of α, β at p form a positively oriented basis and $\varepsilon_p = -1$ otherwise.

An example of a self-homeomorphism of Σ is provided by the Dehn twist τ_α about a simple closed curve $\alpha \subset \Sigma$. It is defined as follows. Identify a regular neighborhood of α in Σ with the cylinder $S^1 \times [0,1]$ so that $\alpha = S^1 \times (1/2)$. We choose this identification so that the product of the counterclockwise orientation on $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ and the right-handed orientation on $[0,1]$ corresponds to the given orientation on Σ. The Dehn twist $\tau_\alpha : \Sigma \to \Sigma$ is the identity map outside $S^1 \times [0,1]$ and sends any $(x,s) \in S^1 \times [0,1]$ to $(e^{2\pi i s}x, s) \in S^1 \times [0,1]$. To compute the action of τ_α in homology, we orient α in an arbitrary way. The effect of τ_α on an oriented curve transversal to α is to insert $\alpha^{\pm 1}$ at each crossing of α with this curve, where ± 1 is the sign of the crossing. Therefore for any $g \in H$,

$$(\tau_\alpha)_*(g) = g + ([\alpha] \cdot g)[\alpha] \tag{1.1}$$

Note that τ_α and its action on H do not depend on the choice of orientation on α.

A similar construction applies to arcs in Σ whose endpoints are punctures. Assume that Σ is obtained from another surface Σ' by puncturing, i.e., by removing a finite set of points lying in $\mathrm{Int}(\Sigma')$. These points will be called *punctures*. Let α be an embedded arc in Σ' whose endpoints are punctures x_1, x_2 and whose interior lies in Σ. One can define the Dehn "half-twist" $\tau_\alpha : \Sigma \to \Sigma$ which is the identity map outside a regular neighborhood of α in Σ' and which exchanges x_1 and x_2. This homeomorphism is obtained by the isotopy of the identity map of Σ' rotating α about

its midpoint to the angle of π in the direction provided by the orientation of Σ. Restricting the resulting homeomorphism of $\Sigma' \to \Sigma'$ to Σ we obtain τ_α. To compute the action of τ_α on $H = H_1(\Sigma)$ we orient α from x_1 to x_2 and associate to α a loop α' in Σ as follows. Choose a point $z \in \alpha$ and for $i = 1, 2$ denote by μ_i the loop in Σ beginning at z and moving along α until coming very closely to x_i, then encircling x_i in the direction determined by the orientation of Σ and finally moving back to z along α. Set $\alpha' = \mu_1^{-1}\mu_2$. The homotopy class of the loop α' on Σ does not depend on the choice of z. The effect of τ_α on an oriented curve transversal to α is to insert $(\alpha')^{\pm 1}$ at each crossing of α with this curve. Thus for any $g \in H$, we have

$$(\tau_\alpha)_*(g) = g + ([\alpha] \cdot g)[\alpha'] \tag{1.2}$$

where $[\alpha] \cdot g = -g \cdot [\alpha] \in \mathbf{Z}$ is the algebraic intersection number of g with the 1-dimensional homology class $[\alpha]$ of Σ "modulo infinity" represented by α.

In general, the action of $\mathrm{Homeo}(\Sigma)$ on $H = H_1(\Sigma)$ is not faithful. We point out one source of non-faithfulness. If $\alpha, \beta \subset \Sigma$ are simple closed curves with $[\alpha] \cdot [\beta] = 0$ then formula (1.1) implies that $(\tau_\alpha)_*$ and $(\tau_\beta)_*$ commute in $\mathrm{Aut}(H)$. The Dehn twists τ_α, τ_β themselves commute if and only if α is isotopic to a simple closed curve disjoint from β, see for instance [Iv1]. It is easy to give examples of simple closed curves $\alpha, \beta \subset \Sigma$ which are not disjoint up to isotopy but have zero algebraic intersection number. Then the commutator $[\tau_\alpha, \tau_\beta]$ lies in the kernel of the homological representation. Using (1.2), one can similarly derive elements of the kernel from embedded arcs with endpoints in punctures.

1.2. Braid groups. Let $D = \{z \in \mathbf{C} \,|\, |z| \leq 1\}$ be the unit disc with counterclockwise orientation. Fix a set of $n \geq 1$ distinct punctures $X = \{x_1, ..., x_n\} \subset \mathrm{Int}(D)$. We shall assume that $x_1, ..., x_n \in (-1, +1) = \mathbf{R} \cap \mathrm{Int}(D)$ and $x_1 < x_2 < ... < x_n$. Set $D_n = D \backslash X$. The group $\mathrm{Homeo}(D_n)$ is denoted B_n and called the *n-th braid group*. An element of B_n is an isotopy class of a homeomorphism $D_n \to D_n$ which fixes $\partial D_n = \partial D = S^1$ pointwise. Such a homeomorphism uniquely extends to a homeomorphism $D \to D$ permuting $x_1, ..., x_n$. This defines a group homomorphism from B_n onto the symmetric group S_n. We can equivalently define B_n at the group of isotopy classes of homeomorphisms $D \to D$ which fix ∂D pointwise and preserve X as a set.

For $i = 1, ..., n-1$, the linear interval $[x_i, x_{i+1}] \subset (-1, +1) \subset \mathbf{R}$ is an embedded arc in D with endpoints in the punctures x_i, x_{i+1}. The corresponding Dehn half-twist $D_n \to D_n$ is denoted by σ_i. It is a classical fact that B_n is generated by $\sigma_1, ..., \sigma_{n-1}$ with defining relations (0.1), (0.2). The image of σ_i in S_n is the permutation $(i, i+1)$.

Another definition of B_n can be given in terms of braids. A (geometric) *braid on n strings* is an n-component one-dimensional manifold $E \subset D \times [0,1]$ such that E meets $D \times \{0,1\}$ orthogonally along the set $(X \times 0) \cup (X \times 1)$ and the projection on $[0,1]$ maps each component of E homeomorphically onto $[0,1]$. The braids are considered up to isotopy in $D \times [0,1]$ constant on the endpoints. The group operation in the set of braids is defined by glueing one braid on the top of the other one and compressing the result into $D \times [0,1]$.

The equivalence between these two definitions of B_n is established as follows. Any homeomorphism $h : D \to D$ which fixes ∂D pointwise is related to the identity map id_D by an isotopy $\{h_s : D \to D\}_{s \in [0,1]}$ such that $h_0 = \mathrm{id}_D$ and $h_1 = h$. If $h(X) = X$ then the set $\cup_{s\in[0,1]}(h_s(X) \times s) \subset D \times [0,1]$ is a braid. Its isotopy class depends only on the element of B_n represented by h. This establishes an isomorphism between B_n and the group of braids on n strings. The generator $\sigma_i \in B_n$ corresponds to the i-th "elementary" braid represented by a plane diagram consisting of n linear intervals which are disjoint except at one intersection point where the i-th interval goes over the $(i+1)$-th interval.

1.3. The Burau representation. Let Λ denote the ring $\mathbf{Z}[t, t^{-1}]$. The Burau representation $B_n \to GL_n(\Lambda)$ sends the i-th generator $\sigma_i \in B_n$ into the matrix

$$I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

where I_k denotes the identity $(k \times k)$-matrix and the non-trivial (2×2)-block appears in the i-th and $(i+1)$-th rows and columns (see [Bu]). Substituting $t = 1$, we obtain the standard representation of the symmetric group S_n by permutation matrices or equivalently the homological action of B_n on $H_1(D_n) = \mathbf{Z}^n$. The Burau representation is reducible: it splits as a direct sum of an $(n-1)$-dimensional representation and the trivial one-dimensional representation.

The Burau representation is known to be faithful for $n \leq 3$, see [Bir]. J. Moody [Mo] proved in 1991 that it is not faithful for $n \geq 9$. D. Long and M. Paton [LP] extended Moody's argument to $n \geq 6$. Recently, S. Bigelow [Bi1] proved that this representation is not faithful for $n = 5$. The case $n = 4$ remains open.

The geometric idea allowing to detect non-trivial elements in the kernel of the Burau representation is parallel to the one at the end of Sect. 1.1. We first give a homological description of the Burau representation. Observe that $H_1(D \backslash x_i) = \mathbf{Z}$ is generated by the class of a small loop encircling x_i in the counterclockwise direction. Each loop in $D \backslash x_i$ represents k times the generator where k is the *winding number* of

the loop around x_i. Consider the homomorphism $H_1(D_n) \to \mathbf{Z}$ sending the homology class of a loop to its *total winding number* defined as the sum of its winding numbers around $x_1, ..., x_n$. Let $\tilde{D}_n \to D_n$ be the corresponding regular covering. The group of covering transformations of $\tilde{D}_n$ is $\mathbf{Z}$ which we write as a multiplicative group with generator t. The group $H_1(\tilde{D}_n)$ acquires thus the structure of a Λ-module. It is easy to check that this is a free Λ-module of rank $n-1$.

Fix a basepoint $d \in \partial D$. Any homeomorphism $h : D_n \to D_n$ representing an element of B_n lifts uniquely to a homeomorphism $\tilde{h} : \tilde{D}_n \to \tilde{D}_n$ which fixes the fiber over d pointwise. This induces a Λ-linear automorphism $\tilde{h}_*$ of $H_1(\tilde{D}_n)$. The map $h \mapsto \tilde{h}_*$ defines a representation $B_n \to \mathrm{Aut}(H_1(\tilde{D}_n))$ equivalent to the $(n-1)$-dimensional Burau representation.

Now we extend the algebraic intersection to arcs and refine it so that it takes values in Λ. Let α, β be two embedded oriented arcs in D_n with endpoints in the punctures. (We assume that all four endpoints of α, β are distinct so that $n \geq 4$). Let $\tilde{\alpha}, \tilde{\beta}$ be lifts of α, β to $\tilde{D}_n$, respectively. Set

$$\langle \alpha, \beta \rangle = \sum_{k \in \mathbf{Z}} (t^k \tilde{\alpha} \cdot \tilde{\beta})\, t^k \in \Lambda$$

where $t^k \tilde{\alpha} \cdot \tilde{\beta} \in \mathbf{Z}$ is the algebraic intersection number of the arcs $t^k \tilde{\alpha}, \tilde{\beta}$ in $\tilde{D}_n$. This finite sum is only defined up to multiplication by a power of t depending on the choice of $\tilde{\alpha}, \tilde{\beta}$. This will not be important for us since we are only interested in whether or not $\langle \alpha, \beta \rangle = 0$. To compute $\langle \alpha, \beta \rangle$ explicitly one deforms α in general position with respect to β. Then $\langle \alpha, \beta \rangle = \sum_{p \in \alpha \cap \beta} \varepsilon_p\, t^{k_p}$ where $\varepsilon_p = \pm$ is the intersection sign at p and $k_p \in \mathbf{Z}$. The exponents $\{k_p\}_p$ are determined by the following condition: if $p, q \in \alpha \cap \beta$, then $k_p - k_q$ is the total winding number of the loop going from p to q along α and then from q to p along β. Note that

$$\langle \beta, \alpha \rangle = \sum_{k \in \mathbf{Z}} (t^k \tilde{\beta} \cdot \tilde{\alpha})\, t^k = \sum_{k \in \mathbf{Z}} (\tilde{\beta} \cdot t^{-k} \tilde{\alpha})\, t^k = -\sum_{k \in \mathbf{Z}} (t^{-k} \tilde{\alpha} \cdot \tilde{\beta})\, t^k = -\overline{\langle \alpha, \beta \rangle}$$

where the overline denotes the involution in Λ sending any t^k to t^{-k}. Hence $\langle \alpha, \beta \rangle = 0$ if and only if $\langle \beta, \alpha \rangle = 0$.

We claim that if $\langle \alpha, \beta \rangle = 0$, then the automorphisms $(\tilde{\tau}_\alpha)_*, (\tilde{\tau}_\beta)_*$ of $H_1(\tilde{D}_n)$ commute. Observe that the loops α', β' on D_n associated to α, β as in Sect. 1.1 have zero total winding numbers and therefore lift to certain loops $\tilde{\alpha}', \tilde{\beta}'$ in $\tilde{D}_n$. The effect of $\tilde{\tau}_\alpha$ on any oriented loop γ in $\tilde{D}_n$ is to insert a lift of $(\alpha')^{\pm 1}$ at each crossing of γ with the preimage of α in $\tilde{D}_n$. Thus

$$(\tilde{\tau}_\alpha)_*([\gamma]) = [\gamma] + \lambda_\gamma\, [\tilde{\alpha}']$$

for a certain Laurent polynomial $\lambda_\gamma \in \Lambda$. The coefficients of λ_γ are the algebraic intersection numbers of γ with lifts of α to $\tilde{D}_n$. By $\langle \alpha, \beta \rangle = 0$, any lift of α has algebraic intersection number zero with any lift of β and hence with any lift of β'. Therefore, $\lambda_{\tilde{\beta}'} = 0$ and $(\tilde{\tau}_\alpha)_*([\tilde{\beta}']) = [\tilde{\beta}']$. Similarly, $(\tilde{\tau}_\beta)_*([\gamma]) = [\gamma] + \mu_\gamma\,[\tilde{\beta}']$ with $\mu_\gamma \in \Lambda$ and $(\tilde{\tau}_\beta)_*([\tilde{\alpha}']) = [\tilde{\alpha}']$. We conclude that

$$(\tilde{\tau}_\alpha \tilde{\tau}_\beta)_*([\gamma]) = [\gamma] + \lambda_\gamma\,[\tilde{\alpha}'] + \mu_\gamma\,[\tilde{\beta}'] = (\tilde{\tau}_\beta \tilde{\tau}_\alpha)_*([\gamma]).$$

To show that the Burau representation is not faithful it remains to provide an example of oriented embedded arcs α, β in D_n with endpoints in distinct punctures such that $\langle \alpha, \beta \rangle = 0$ and $\tau_\alpha \tau_\beta \neq \tau_\beta \tau_\alpha$ in B_n. For $n \geq 6$, the simplest known example (see [Bi1]) is provided by the pair $\alpha = \varphi_1([x_3, x_4])$, $\beta = \varphi_2([x_3, x_4])$ where

$$\varphi_1 = \sigma_1^2 \sigma_2^{-1} \sigma_5^{-2} \sigma_4, \quad \varphi_2 = \sigma_1^{-1} \sigma_2 \sigma_5 \sigma_4^{-1}.$$

To compute $\langle \alpha, \beta \rangle$ one can draw α, β and use the recipe above. To prove that the braids $\tau_\alpha = \varphi_1 \sigma_3 \varphi_1^{-1}$ and $\tau_\beta = \varphi_2 \sigma_3 \varphi_2^{-1}$ do not commute, one can use the solution of the word problem in B_n or the methods of the Thurston theory of surfaces (cf. Sect. 2.2). Thus the commutator $[\tau_\alpha, \tau_\beta]$ lies in the kernel of the Burau representation. This commutator can be represented by a word of length 44 in the generators $\sigma_1, ..., \sigma_5$.

Similar ideas apply in the case $n = 5$, although one has to extend them to arcs relating the punctures to the base point $d \in \partial D$. The shortest known word in the generators $\sigma_1, ..., \sigma_4$ representing an element of the kernel has length 120.

2. THE WORK OF BIGELOW

2.1. A representation of B_n. We use the notation $D, D_n, X = \{x_1, ..., x_n\}$ introduced in Sect. 1. Let C be the space of all unordered pairs of distinct points in D_n. This space is obtained from $(D_n \times D_n)\backslash$diagonal by the identification $\{x, y\} = \{y, x\}$ for any distinct $x, y \in D_n$. It is clear that C is a connected non-compact 4-manifold with boundary. It has a natural orientation induced by the counterclockwise orientation of D_n. Set $d = -i \in \partial D$ where $i = \sqrt{-1}$ and $d' = -i\, e^{\frac{\varepsilon \pi i}{2}} \in \partial D$ with small positive ε. We take $c_0 = \{d, d'\}$ as the basepoint for C.

A closed curve $\alpha : [0, 1] \to C$ can be written in the form $\alpha(s) = \{\alpha_1(s), \alpha_2(s)\}$ where $s \in [0, 1]$ and α_1, α_2 are arcs in D_n such that $\{\alpha_1(0), \alpha_2(0)\} = \{\alpha_1(1), \alpha_2(1)\}$. The arcs α_1, α_2 are either both loops or can be composed with each other. They form thus a closed oriented one-manifold mapped to D_n. Let $a(\alpha) \in \mathbf{Z}$ be the total winding

number of this one-manifold around the punctures $\{x_1, ..., x_n\}$. Composing the map $s \mapsto (\alpha_1(s) - \alpha_2(s))/|\alpha_1(s) - \alpha_2(s)| : [0,1] \to S^1$ with the projection $S^1 \to \mathbf{RP}^1$ we obtain a loop in RP^1. The corresponding element of $H_1(RP^1) = \mathbf{Z}$ is denoted by $b(\alpha)$. The formula $\alpha \mapsto q^{a(\alpha)} t^{b(\alpha)}$ defines a homomorphism, ϕ, from $H_1(C)$ to the (multiplicatively written) free abelian group with basis q, t. Let $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ be the group ring of this group.

Let $\tilde{C} \to C$ be a regular covering corresponding to the kernel of ϕ. The generators q, t act on $\tilde{C}$ as commuting covering transformations. The homology group $H_2(\tilde{C}) = H_2(\tilde{C}; \mathbf{Z})$ becomes thus an R-module.

Any self-homeomorphism h of D_n induces by $h(\{x, y\}) = \{h(x), h(y)\}$ a homeomorphism $C \to C$ also denoted h. It is easy to check that $h(c_0) = c_0$ and the action of h on $H_1(C)$ commutes with ϕ. Therefore this homeomorphism $C \to C$ lifts uniquely to a map $\tilde{h} : \tilde{C} \to \tilde{C}$ which fixes the fiber over c_0 pointwise and commutes with the covering transformations. Consider the representation $B_n \to \mathrm{Aut}(H_2(\tilde{C}))$ sending the isotopy class of h to the R-linear automorphism $\tilde{h}_*$ of $H_2(\tilde{C})$.

2.2. Theorem. (S. Bigelow [Bi2]) – *The representation $B_n \to \mathrm{Aut}(H_2(\tilde{C}))$ is faithful for all $n \geq 1$.*

We outline below the main ideas of Bigelow's proof. The proof uses almost no information about the structure of the R-module $H_2(\tilde{C})$. The only thing needed is the absence of R-torsion or more precisely the fact that multiplication by a non-zero polynomial of type $q^a t^b - 1$ has zero kernel in $H_2(\tilde{C})$. In fact, $H_2(\tilde{C})$ is a free R-module of rank $n(n-1)/2$, as it was essentially shown in [La].

We shall use one well-known fact concerning isotopies of arcs on surfaces. Let N, T be embedded arcs in D_n with distinct endpoints lying either in the punctures or on ∂D_n. Assume that the interiors of N, T do not meet ∂D_n, and that N intersects T transversely (in a finite number of points). A *bigon* for the pair (N, T) is an embedded disc in $\mathrm{Int}(D_n)$ whose boundary is formed by one subarc of N and one subarc of T and whose interior is disjoint from N and T. It is clear that in the presence of a bigon there is an isotopy of T constant on the endpoints and decreasing $\#(N \cap T)$ by two. Thurston's theory of surfaces implies the converse: if there is an isotopy of T (rel endpoints) decreasing $\#(N \cap T)$ then the pair (N, T) has at least one bigon, cf. [FLP, Prop. 3.10].

2.3. Noodles and forks. We need the following notation. For arcs $\alpha, \beta : [0,1] \to D_n$ such that $\alpha(s) \neq \beta(s)$ for all $s \in [0,1]$, we denote by $\{\alpha, \beta\}$ the arc in C given by

$\{\alpha,\beta\}(s)=\{\alpha(s),\beta(s)\}$. We fix once and forever a point $\tilde{c}_0\in\tilde{C}$ lying over $c_0\in C$.

A *noodle* in D_n is an embedded arc $N\subset D_n$ with endpoints d and d'. For a noodle N, the set $\Sigma_N=\{\{x,y\}\in C\,|\,x,y\in N, x\neq y\}$ is a surface in C containing c_0. This surface is homeomorphic to a triangle with one edge removed. We orient N from d to d' and orient Σ_N as follows: at a point $\{x,y\}=\{y,x\}\in\Sigma_N$ such that x is closer to d along N than y, the orientation of Σ_N is the product of the orientations of N at x and y in this order. Let $\tilde{\Sigma}_N$ be the lift of Σ_N to $\tilde{C}$ containing $\tilde{c}_0$. The orientation of Σ_N lifts to $\tilde{\Sigma}_N$ in the obvious way. Clearly, $\tilde{\Sigma}_N$ is a proper surface in $\tilde{C}$ in the sense that $\tilde{\Sigma}_N\cap\partial\tilde{C}=\partial\tilde{\Sigma}_N$.

A *fork* in D_n is an embedded tree $F\subset D$ formed by three edges and four vertices d,x_i,x_j,z such that $F\cap\partial D=d, F\cap X=\{x_i,x_j\}$ and z is a common vertex of all 3 edges. The edge, H, relating d to z is called the *handle* of F. The union, T, of the other two edges is an embedded arc with endpoints $\{x_i,x_j\}$. This arc is called the *tines* of F. Note that in a small neighborhood of z, the handle H lies on one side of T which distinguishes a side of T. We orient T so that its distinguished side lies on its right. The handle H also has a distinguished side determined by d'. Pushing slightly the graph $F=T\cup H$ to the distinguished side (fixing the vertices x_i,x_j and pushing d to d') we obtain a "parallel copy" $F'=T'\cup H'$. The graph F' is a fork with handle H', tines T', and vertices d',x_i,x_j,z' where $z'=T'\cap H'$ lies on the distinguished side of both T and H. We can assume that F' meets F only in common vertices $\{x_i,x_j\}=T\cap T'$ and in one point lying on $H\cap T'$ close to z,z'.

For a fork F, the set $\Sigma_F=\{\{y,y'\}\in C\,|\,y\in T\backslash\{x_i,x_j\}, y'\in T'\backslash\{x_i,x_j\}\}$ is a surface in C homeomorphic to $(0,1)^2$. Let α_0 be an arc from d to z along H and let α_0' be an arc from d' to z' along H'. Consider the arc $\{\alpha_0,\alpha_0'\}$ in C and denote by $\tilde{\alpha}$ its lift to $\tilde{C}$ which starts in $\tilde{c}_0$. Let $\tilde{\Sigma}_F$ be the lift of Σ_F to $\tilde{C}$ which contains the lift $\tilde{\alpha}(1)$ of the point $\{z,z'\}\in\Sigma_F$. The surfaces Σ_F and $\tilde{\Sigma}_F$ have a natural orientation determined by the orientation of T and the induced orientation of T'.

We shall use the surfaces $\tilde{\Sigma}_N,\tilde{\Sigma}_F$ to establish a duality between noodles and forks. More precisely, for any noodle N and any fork F we define an element $\langle N,F\rangle$ of R as follows. By applying a preliminary isotopy we can assume that N intersects T transversely in $m\geq 0$ points $z_1,...,z_m$ (the numeration is arbitrary; the intersection of N with H may be not transversal). We choose the parallel fork $F'=T'\cup H'$ as above so that T' meets N transversely in m points $z_1',...,z_m'$ where each pair z_i,z_i' is joined by a short arc in N which lies in the narrow strip bounded by $T\cup T'$ and meets no other z_j,z_j'. Then the surfaces Σ_F and Σ_N intersect transversely in m^2 points $\{z_i,z_j'\}$ where $i,j=1,...,m$. Therefore for any $a,b\in\mathbf{Z}$, the image $q^at^b\tilde{\Sigma}_N$

of $\tilde{\Sigma}_N$ under the covering transformation $q^a t^b$ meets $\tilde{\Sigma}_F$ transversely. Consider the algebraic intersection number $q^a t^b \tilde{\Sigma}_N \cdot \tilde{\Sigma}_F \in \mathbf{Z}$ and set

$$\langle N, F\rangle = \sum_{a,b \in \mathbf{Z}} (q^a t^b \tilde{\Sigma}_N \cdot \tilde{\Sigma}_F)\, q^a t^b.$$

The sum on the right-hand side is finite (it has $\leq m^2$ terms) and thus defines an element of R.

2.4. Lemma. – *$\langle N, F\rangle$ is invariant under isotopies of N and F in D_n constant on the endpoints.*

Proof. – We first compute $\langle N, F\rangle$ explicitly. Let $N \cap T = \{z_1, ..., z_m\}$ and $N \cap T' = \{z'_1, ..., z'_m\}$ as above. For every pair $i, j \in \{1, ..., m\}$, there exist unique integers $a_{i,j}, b_{i,j} \in \mathbf{Z}$ such that $q^{a_{i,j}} t^{b_{i,j}} \tilde{\Sigma}_N$ intersects $\tilde{\Sigma}_F$ at a point lying over $\{z_i, z'_j\} \in C$. Let $\varepsilon_{i,j} = \pm 1$ be the sign of that intersection. Then

$$\langle N, F\rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} \varepsilon_{i,j}\, q^{a_{i,j}} t^{b_{i,j}}. \tag{2.1}$$

The numbers $a_{i,j}, b_{i,j}$ can be computed as follows. Let α_0 be an arc from d to z along H and let α'_0 be an arc from d' to z' along H'. Let β_i be an arc from z to z_i along T and let β'_j be an arc from z' to z'_j along T'. Finally, let $\gamma_{i,j}$ and $\gamma'_{i,j}$ be disjoint arcs in N connecting the points z_i, z'_j to the endpoints of N. Note that $\delta_{i,j} = \{\alpha_0, \alpha'_0\}\{\beta_i, \beta'_j\}\{\gamma_{i,j}, \gamma'_{i,j}\}$ is a loop in C. Then

$$q^{a_{i,j}} t^{b_{i,j}} = \phi(\delta_{i,j}). \tag{2.2}$$

Indeed, we can lift $\delta_{i,j}$ to a path $\tilde{\alpha}\tilde{\beta}\tilde{\gamma}$ in $\tilde{C}$ beginning at $\tilde{c}_0$ where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are lifts of $\{\alpha_0, \alpha'_0\}, \{\beta_i, \beta'_j\}, \{\gamma_{i,j}, \gamma'_{i,j}\}$, respectively. By definition of $\tilde{\Sigma}_F$, the point $\tilde{\alpha}(1) = \tilde{\beta}(0)$ lies on $\tilde{\Sigma}_F$. Hence the lift $\tilde{\beta}$ of $\{\beta_i, \beta'_j\}$ lies on $\tilde{\Sigma}_F$. The path $\tilde{\alpha}\tilde{\beta}\tilde{\gamma}$ ends at $\phi(\delta_{i,j})(\tilde{c}_0) = \tilde{\gamma}(1)$. Hence the lift $\tilde{\gamma}$ of $\{\gamma_{i,j}, \gamma'_{i,j}\}$ lies on $\phi(\delta_{i,j})\tilde{\Sigma}_N$. Therefore the point $\tilde{\beta}(1) = \tilde{\gamma}(0)$ lying over $\{z_i, z'_j\}$ belongs to both $\tilde{\Sigma}_F$ and $\phi(\delta_{i,j})\tilde{\Sigma}_N$. This yields (2.2).

Note that the residue $b_{i,j} \pmod 2$ depends on whether the two points of D_n forming a point in C switch places moving along the loop $\delta_{i,j}$. This is determined by which of z_i, z'_j lies closer to d along N.

To compute $\varepsilon_{i,j}$ we observe that $\varepsilon_{i,j}$ is determined by the signs of the intersections of N and T at z_i, z_j and by which of z_i, z'_j lies closer to d on N. The sign of the

intersection of N and T at z_i is $+$ if N crosses T from the left to the right and is $-$ otherwise. By our choice of orientations on N and T, this sign is $+$ if z_i lies closer to d along N than z_i' and is $-$ otherwise. Hence, this sign is $(-1)^{b_{i,i}}$.Therefore $\varepsilon_{i,j}$ is determined by $b_{i,i}+b_{j,j}+b_{i,j} \pmod 2$. A precise computation shows that

$$\varepsilon_{i,j} = -(-1)^{b_{i,i}+b_{j,j}+b_{i,j}}. \tag{2.3}$$

Now we can prove the lemma. It suffices to fix F and to prove that $\langle N, F\rangle$ is invariant under isotopies of N. A generic isotopy of N in D_n can be split into a finite sequence of local moves of two kinds: (i) isotopies keeping N transversal to T, (ii) a move pushing a small subarc of N across a subarc of $T\backslash z$. It is clear from the discussion above that the move (i) does not change $\langle N, F\rangle$. The move (ii) adds two new intersection points z_{m+1}, z_{m+2} to the set $N\cap T = \{z_1, ..., z_m\}$. It follows from definitions and the discussion above that for any $i = 1, ..., m+2$,

$$a_{i,m+1} = a_{i,m+2}, \ \ b_{i,m+1} = b_{i,m+2}, \ \ \varepsilon_{i,m+1} = -\varepsilon_{i,m+2}.$$

Hence for any $i = 1, ..., m+2$, the terms $\varepsilon_{i,m+1} q^{a_{i,m+1}} t^{b_{i,m+1}}$ and $\varepsilon_{i,m+2} q^{a_{i,m+2}} t^{b_{i,m+2}}$ cancel each other. Similarly, for any $j = 1, ..., m$, the terms $\varepsilon_{i,j} q^{a_{i,j}} t^{b_{i,j}}$ with $i = m+1, m+2$ cancel each other. Therefore $\langle N, F\rangle$ is the same before and after the move.

2.5. Lemma. – *$\langle N, F\rangle = 0$ if and only if there is an isotopy $\{T(s)\}_{s\in[0,1]}$ of the tines $T = T(0)$ of F in D_n (rel endpoints) such that $T(1)$ is disjoint from N.*

Proof. – Any isotopy $\{T(s)\}_{s\in[0,1]}$ of $T = T(0)$ extends to an ambient isotopy of D_n constant on ∂D_n and therefore extends to an isotopy $\{F(s)\}_{s\in[0,1]}$ of the fork $F = F(0)$. If $T(1)$ is disjoint from N then by Lemma 2.4, $\langle N, F\rangle = \langle N, F(1)\rangle = 0$.

The hard part of the lemma is the opposite implication. By applying a preliminary isotopy, we can assume that T intersects N transversely at a *minimal* number of points $z_1, ..., z_m$ with $m \geq 0$. We assume that $m \geq 1$ and show that $\langle N, F\rangle \neq 0$. To this end we use the lexicographic ordering on monomials $q^a t^b$. Namely we write $q^a t^b \geq q^{a'} t^{b'}$ with $a, b, a', b' \in \mathbf{Z}$ if either $a > a'$ or $a = a'$ and $b \geq b'$. We say that the ordered pair (i, j) with $i, j \in \{1, ..., m\}$ is *maximal* if $q^{a_{i,j}} t^{b_{i,j}} \geq q^{a_{k,l}} t^{b_{k,l}}$ for any $k, l \in \{1, ..., m\}$. We claim that

$(*)$ *if the pair (i, j) is maximal, then $b_{i,i} = b_{j,j} = b_{i,j}$.*

This claim and (2.3) imply that all entries of the maximal monomial, say $q^a t^b$, in (2.1) occur with the same sign $-(-1)^b$. Hence $\langle N, F\rangle \neq 0$.

To prove $(*)$ we first compute $a_{i,j}$ for any i, j (not necessarily maximal). Let ξ_i be the loop obtained by moving from d to z_i along F then back to d along N. Let a_i be the total winding number of ξ_i around all n punctures. Let ξ be the loop obtained by moving from d to d' along N, and then moving clockwise along ∂D back to d. Let a be the total winding number of ξ. We claim that

$$a_{i,j} = a_i + a_j + a. \tag{2.4}$$

Indeed, if $b_{i,j}$ is even then the paths $\alpha_0\beta_i\gamma_{i,j}$ and $\alpha'_0\beta'_j\gamma'_{i,j}$ (in the notation of Lemma 2.4) are loops and $a_{i,j}$ is the sum of their total winding numbers. These loops are homotopic in D_n to ξ_i and $\xi_j\xi$, respectively. This implies (2.4). If $b_{i,j}$ is odd then $a_{i,j}$ is the total winding number of the loop $\alpha_0\beta_i\gamma_{i,j}\alpha'_0\beta'_j\gamma'_{i,j}$. This loop is homotopic in D_n to $\xi_i\xi\xi_j$ which implies (2.4) in this case.

Suppose now that the pair (i, j) is maximal. Then $a_{i,j}$ is maximal among all the integers $a_{k,l}$. By (2.4), it follows that $a_i = a_j$ is maximal among all the integers a_k. (Although we shall not need it, observe that then $a_{i,i} = a_{j,j} = a_{i,j}$).

We now show that $b_{i,i} = b_{i,j}$. By the maximality of (i, j), we have that $b_{i,i} \leq b_{i,j}$. Suppose, seeking a contradiction, that $b_{i,i} < b_{i,j}$. Let α be an embedded arc from z'_i to z'_j along T'. Let β be an embedded arc from z'_j to z'_i along N.

If β does not pass through the point z_i, then we denote by w the winding number of the loop $\alpha\beta$ around z_i. Observe that $b_{i,j} - b_{i,i} = 2w$. To see this, consider the loop $\delta_{i,j} = \{\alpha_0, \alpha'_0\}\{\beta_i, \beta'_j\}\{\gamma_{i,j}, \gamma'_{i,j}\}$ appearing in (2.2). Clearly, $\beta'_j \sim \beta'_i\alpha$, where $\sim$ denotes homotopy of paths in $D_n \backslash z_i$ constant on the endpoints. The assumption that β does not pass through z_i implies that $\gamma_{i,j} = \gamma_{i,i}$ and $\gamma'_{i,j} \sim \beta\gamma'_{i,i}$. Then

$$\delta_{i,j} = \{\alpha_0, \alpha'_0\}\{\beta_i, \beta'_j\}\{\gamma_{i,j}, \gamma'_{i,j}\} \sim \{\alpha_0, \alpha'_0\}\{\beta_i, \beta'_i\}\{z_i, \alpha\beta\}\{\gamma_{i,i}, \gamma'_{i,i}\}$$

where z_i denotes the constant path in z_i. This implies that $b_{i,j} - b_{i,i} = 2w$.

If β passes through z_i, we first modify β in a small neighborhood of z_i so that z_i lies to its left. Let w be the winding number of the loop $\alpha\beta$ around z_i. A little more difficult but similar argument shows that $b_{i,j} - b_{i,i} = 2w - 1$. In either case $w > 0$.

Let $D_0 = D \backslash z_i$ and $p : \hat{D}_0 \to D_0$ be the universal (infinite cyclic) covering. Let $\hat{\alpha}$ be a lift of α to $\hat{D}_0$. Let $\hat{\beta}$ be the lift of β to $\hat{D}_0$ which starts at $\hat{\beta}(1)$. Consider a small neighborhood $V \subset D$ of the short arc in N connecting z_i to z'_i such that V meets $\alpha\beta$ only at z'_i. Let γ be a generic loop in $V \backslash z_i$ based at z'_i which winds w times around z_i in the clockwise direction. Let $\hat{\gamma}$ be the lift of γ to $\hat{D}_0$ beginning at $\hat{\beta}(1)$ and ending at $\hat{\alpha}(0)$. We can assume that $\hat{\gamma}$ is an embedded arc meeting $\hat{\alpha}\hat{\beta}$ only at

the endpoints. Let $\hat{z}'_k$ be the first point of $\hat{\alpha}$ which lies also on $\hat{\beta}$ (possibly $\hat{z}'_k = \hat{\alpha}(1)$). Then $p(\hat{z}'_k) = z'_k$ for some $k = 1, ..., n$. Let $\hat{\alpha}'$ be the initial segment of $\hat{\alpha}$ ending at $\hat{z}'_k$. Let $\hat{\beta}'$ be the final segment of $\hat{\beta}$ starting at $\hat{z}'_k$. Set $\hat{\delta} = \hat{\alpha}'\hat{\beta}'\hat{\gamma}$. It is clear that $\hat{\delta}$ is a simple closed curve in $\hat{D}_0$. By the Jordan curve theorem it bounds a disc, $B \subset \hat{D}_0$, which lies either on the left or on the right of $\hat{\gamma}$. Since γ passes clockwise around z_i, the component of $D_0 \backslash \mathrm{Im}(\gamma)$ adjacent to z_i lies on the right of γ. The set $p(B) \subset D_0$ being compact can not pass non-trivially over this component. Hence B lies on the left of $\hat{\gamma}$. Therefore $\hat{\delta}$ passes counterclockwise around B.

The number $a_k - a_i$ is equal to the total winding number of the loop $p(\hat{\alpha}')p(\hat{\beta}')$ in D_n around the punctures $x_1, ..., x_n$. Since γ is contractible in D_n, the loop $p(\hat{\alpha}')p(\hat{\beta}')$ is homotopic in D_n to $p(\hat{\alpha}')p(\hat{\beta}')\gamma = p(\hat{\delta})$. Therefore $a_k - a_i$ is equal to the number of points in $B \cap p^{-1}(X)$. Since a_i is maximal, we must have $a_k = a_i$ and $B \cap p^{-1}(X) = \emptyset$, so that $p(B) \subset D_n \backslash z_i$. Then we can isotop T so as to have fewer points of intersection with N. To see this we shall construct a bigon for the pair (N, T). If B meets $p^{-1}(N \cup T)$ only along $\hat{\alpha}'\hat{\beta}'$ then the projection $p|_{\mathrm{Int}(B)} : \mathrm{Int}(B) \to D_n \backslash z_i$ must be injective. It follows that $w = 1$ and the union of $p(B)$ with the small disc bounded by γ in V is a bigon for (N, T). Assume that $\mathrm{Int}(B) \cap p^{-1}(N \cup T) \neq \emptyset$. Note that $p^{-1}(N)$ (resp. $p^{-1}(T)$) is an embedded one-manifold in $\hat{D}_0$ whose components are non-trivially permuted by any covering transformation. If $p^{-1}(N)$ intersects $\mathrm{Int}(B)$ then this intersection consists of a finite set of disjoint arcs with endpoints on $\hat{\alpha}'$. At least one of this arcs bounds together with a subarc of $\hat{\alpha}'$ a disc, $B_0 \subset B$, whose interior does not meet $p^{-1}(N)$. If $p^{-1}(N)$ does not meet $\mathrm{Int}(B)$ then we set $B_0 = B$. Applying the same construction to the intersection of B_0 with $p^{-1}(T)$ we obtain a bigon $B_{00} \subset B_0$ for the pair $(p^{-1}(N), p^{-1}(T))$. The restriction of p to B_{00} is injective and yields a bigon for (N, T). Hence the intersection $N \cap T$ is not minimal. This contradicts our choice of N, T. Therefore, the assumption $b_{i,i} < b_{i,j}$ must have been false. So, $b_{i,i} = b_{i,j}$. Similarly, $b_{j,j} = b_{i,j}$. This completes the proof of $(*)$ and of Lemma 2.5.

2.6. Lemma. – *If a self-homeomorphism h of D_n represents an element of* $\mathrm{Ker}(B_n \to \mathrm{Aut}(H_2(\tilde{C})))$ *then for any noodle N and any fork F, we have $\langle N, h(F)\rangle = \langle N, F\rangle$.*

Proof. – Let $\{U_i \subset \mathrm{Int}(D)\}_{i=1}^m$ be disjoint closed disc neighborhoods of the points $\{x_i\}_{i=1}^m$, respectively. Let U be the set of points $\{x, y\} \in C$ such that at least one of x, y lies in $\cup_{i=1}^m U_i$. Let $\tilde{U} \subset \tilde{C}$ be the preimage of U under the covering map $\tilde{C} \to C$. Observe that the surface $\tilde{\Sigma}_F$ is an open square such that for a sufficiently

big concentric closed subsquare $S \subset \tilde{\Sigma}_F$ we have $\tilde{\Sigma}_F \backslash S \subset \tilde{U}$. Hence $\tilde{\Sigma}_F$ represents a relative homology class $[\tilde{\Sigma}_F] \in H_2(\tilde{C}, \tilde{U})$. The boundary homomorphism $H_2(\tilde{C}, \tilde{U}) \to H_1(\tilde{U})$ maps $[\tilde{\Sigma}_F]$ into $[\partial S] \in H_1(\tilde{U})$. A direct computation in $\pi_1(U)$ (see [Bi2]) shows that $(q-1)^2(qt+1)[\partial S] = 0$. Therefore $(q-1)^2(qt+1)[\tilde{\Sigma}_F] = j(v_F)$ where j is the inclusion homomorphism $H_2(\tilde{C}) \to H_2(\tilde{C}, \tilde{U})$ and $v_F \in H_2(\tilde{C})$. Deforming if necessary N, we can assume that $N \cap (\cup_i U_i) = \emptyset$. Then $\tilde{\Sigma}_N \cap \tilde{U} = \emptyset$ and therefore

$$(q-1)^2(qt+1)\langle N, F\rangle = \sum_{a,b \in \mathbf{Z}} (q^a t^b \tilde{\Sigma}_N \cdot v_F)\, q^a t^b$$

where $q^a t^b \tilde{\Sigma}_N \cdot v_F$ is the (well-defined) algebraic intersection number between a properly embedded surface and a 2-dimensional homology class. (This number does not depend on the choice of v_F as above).

Any self-homeomorphism h of D_n is isotopic to a self-homeomorphism of D_n preserving the set U. Therefore $v_{h(F)} = \tilde{h}_*(v_F)$. If $\tilde{h}_* = \mathrm{id}$, then

$$q^a t^b \tilde{\Sigma}_N \cdot v_F = q^a t^b \tilde{\Sigma}_N \cdot \tilde{h}_*(v_F) = q^a t^b \tilde{\Sigma}_N \cdot v_{h(F)}.$$

This implies that $(q-1)^2(qt+1)\langle N, F\rangle = (q-1)^2(qt+1)\langle N, h(F)\rangle$ and therefore $\langle N, F\rangle = \langle N, h(F)\rangle$.

2.7. Deduction of Theorem 2.2 from the lemmas. – We shall prove that a self-homeomorphism h of D_n representing an element of $\mathrm{Ker}(B_n \to \mathrm{Aut}(H_2(\tilde{C})))$ is isotopic to the identity map rel ∂D_n. We begin with the following assertion.

$(**)$ *An embedded arc T in D_n with endpoints in (distinct) punctures can be isotopped off a noodle N if and only if $h(T)$ can be isotopped off N.*

Indeed, we can extend T to a fork F so that T is the tines of F. By Lemma 2.6, $\langle N, h(F)\rangle = 0$ if and only if $\langle N, F\rangle = 0$. Now Lemma 2.5 implies $(**)$.

We shall apply $(**)$ to the following arcs and noodles. For $i = 1, ..., n-1$, denote by T_i the embedded arc $[x_i, x_{i+1}] \subset (-1, +1) \subset D$ and denote by N_i the i-th "elementary" noodle obtained by rushing from d towards x_i, encircling x_i in the clockwise direction and then moving straight to d'. It is clear that T_i can be isotopped off N_j if and only if $j \neq i, i+1$. This and $(**)$ imply that h induces the identity permutation on the punctures of D_n.

Since T_1 is disjoint from N_3, we can isotop h rel ∂D_n so that $h(T_1)$ is disjoint from N_3. Similarly, $h(T_1)$ can be made disjoint from N_4. As it was explained in

Sect. 2.2, this can be done by a sequence of isotopies eliminating bigons for the pair $(N_4, h(T_1))$. Since N_4 and $h(T_1)$ do not meet N_3, neither do these bigons. Hence our isotopies do not create intersections of $h(T_1)$ with N_3. Repeating this argument, we can assume that $h(T_1)$ is disjoint from all N_i with $i = 3, 4, ..., n-1$. By applying one final isotopy we can make $h(T_1) = T_1$. Applying the same procedure to T_2 we can ensure that $h(T_2) = T_2$ while keeping $h(T_1) = T_1$. Continuing in this way, we can assume that $h(T_i) = T_i$ for all $i = 1, ..., n-1$. Such a homeomorphism h is isotopic to a k-th power ($k \in \mathbf{Z}$) of the Dehn twist about a circle in $\mathrm{Int}(D_n)$ going very closely to ∂D_n. This Dehn twist acts on $H_2(\tilde{C})$ by multiplication by $q^{2n}t^2$. Since by assumption h acts trivially on $H_2(\tilde{C})$, we must have $k = 0$ so that h is isotopic to the identity rel ∂D_n.

2.8. Remarks. – The proof of Lemma 2.6 shows that each fork F determines (a priori non-uniquely) a certain homology class $v_F \in H_2(\tilde{C})$. It follows from the computations in [Bi2] that this class is in fact well-determined by F. Thus, the forks yield a nice geometric way of representing elements of $H_2(\tilde{C})$ (this was implicit in [Kr1]). For instance, for any $1 \leq i < j \leq n$ we can consider the fork consisting of three linear segments connecting the point $-\sqrt{-1}/2$ to d, x_i, x_j. The corresponding classes $\{v_{i,j} \in H_2(\tilde{C})\}_{i,j}$ form a basis of the free R-module $H_2(\tilde{C})$. The action of the braid generators $\sigma_1, ..., \sigma_{n-1}$ on this basis can be described by explicit formulas (see [Bi2], cf. Sect. 3.1).

3. THE WORK OF KRAMMER

3.1. A representation of B_n. Following [Kr2], we denote by $\mathrm{Ref} = \mathrm{Ref}_n$ the set of pairs of integers (i, j) such that $1 \leq i < j \leq n$. Clearly, $\mathrm{card}(\mathrm{Ref}) = n(n-1)/2$.

Let R be a commutative ring with unit and $q, t \in R$ be two invertible elements. Let $V = \oplus_{s \in \mathrm{Ref}} R x_s$ be the free R-module of rank $n(n-1)/2$ with basis $\{x_s\}_{s \in \mathrm{Ref}}$. Krammer [Kr2] defines an R-linear action of B_n on V by

$$\sigma_k(x_{i,j}) = \begin{cases} x_{i,j} & \text{if } k < i-1 \text{ or } j < k, \\ x_{i-1,j} + (1-q)x_{i,j} & \text{if } k = i-1, \\ tq(q-1)x_{i,i+1} + qx_{i+1,j} & \text{if } k = i < j-1, \\ tq^2 x_{i,j} & \text{if } k = i = j-1, \\ x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1} & \text{if } i < k < j-1, \\ x_{i,j-1} + tq^{j-i}(q-1)x_{j-1,j} & \text{if } k = j-1, \\ (1-q)x_{i,j} + qx_{i,j+1} & \text{if } k = j, \end{cases}$$

where $1 \leq i < j \leq n$ and $k = 1, ..., n-1$. That the action of σ_k is invertible and

that relations (0.1), (0.2) are satisfied should be verified by a direct computation. For $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$, this representation is equivalent to the one considered in Sect. 2. In terms of the basis $\{v_{i,j} \in H_2(\tilde{C})\}_{i,j}$ mentioned in Sect. 2.8, the equivalence is given by

$$v_{i,j} = x_{i,j} + (1-q) \sum_{i<k<j} x_{k,j}, \quad x_{i,j} = v_{i,j} + (q-1) \sum_{i<k<j} q^{k-1-i}\, v_{k,j}.$$

3.2. Theorem. (D. Krammer [Kr2]) – *Let* $R = \mathbf{R}[t^{\pm 1}]$, $q \in \mathbf{R}$, *and* $0 < q < 1$. *Then the representation* $B_n \to \mathrm{Aut}(V)$ *defined in Sect. 3.1 is faithful for all* $n \geq 1$.

This Theorem implies Theorem 2.2: if a representation over $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ becomes faithful after assigning a real value to q, then it is faithful itself.

Below we outline the main ideas of Krammer's proof.

3.3. Positive braids and the set $\Omega \subset B_n$. We recall a few facts about the braid group B_n, see [Ch], [Ga], [Mi]. For $i = 1, 2, ..., n-1$ denote by s_i the transposition $(i, i+1) \in S_n$. The set $\{s_1, ..., s_{n-1}\}$ generates the symmetric group S_n. Let $|\ | : S_n \to \mathbf{Z}$ be the length function with respect to this generating set: for $x \in S_n$, $|x|$ is the smallest natural number k such that x is a product of k elements of the set $\{s_1, ..., s_{n-1}\}$. The canonical projection $B_n \to S_n$ has a unique set-theoretic section $r : S_n \to B_n$ such that $r(s_i) = \sigma_i$ for $i = 1, ..., n-1$ and $r(xy) = r(x)\, r(y)$ whenever $|xy| = |x| + |y|$. The group B_n admits a presentation by generators $\{r(x) \,|\, x \in S_n\}$ and relations $r(xy) = r(x)\, r(y)$ for all $x, y \in S_n$ such that $|xy| = |x| + |y|$. Set $\Omega = r(S_n) \subset B_n$. Note that $1 = r(1) \in \Omega$ and $\sigma_i = r(s_i) \in \Omega$ for all i.

The *positive braid monoid* B_n^+ is the submonoid of B_n generated by $\{\sigma_1, ..., \sigma_{n-1}\}$. The elements of B_n^+ are called *positive braids*. Clearly, $\Omega \subset B_n^+$.

For any $x \in B_n^+$ there is a unique longest $x' \in S_n$ such that $x \in r(x')B_n^+$. We denote $r(x') \in \Omega$ by $LF(x)$ where LF stands for the leftmost factor. Observe that

$$LF(xy) = LF(x\, LF(y)) \tag{3.1}$$

for any $x, y \in B_n^+$. This implies that the map $B_n^+ \times \Omega \to \Omega$ defined by $(x, y) \mapsto LF(xy)$ is an action of the monoid B_n^+ on Ω.

3.4. Half-permutations. A set $A \subset \mathrm{Ref}$ is called a *half-permutation* if whenever $(i,j), (j,k) \in A$, one has $(i,k) \in A$. Each half-permutation A determines an ordering $<_A$ on the set $\{1, 2, ..., n\}$ by $i <_A j \Leftrightarrow (i,j) \in A$ (and vice versa).

To state deeper properties of half-permutations we consider the set 2^{Ref} of all subsets of Ref and define a map $L : S_n \to 2^{\mathrm{Ref}}$ by

$$L(x) = \{(i,j) \mid 1 \leq i < j \leq n,\ x^{-1}(i) > x^{-1}(j)\} \subset \mathrm{Ref}.$$

Note that the set $L(x)$ is a half-permutation and $\mathrm{card}(L(x)) = |x|$. It is obvious that the map $L : S_n \to 2^{\mathrm{Ref}}$ is injective.

The key property of half-permutations is the following assertion ([Kr2, Lemma 4.3]): for every half-permutation $A \subset \mathrm{Ref}$ there is a greatest set $A' \subset A$ (with respect to inclusion) such that $A' = L(x)$ for a certain $x \in S_n$. The corresponding braid $r(x) \in \Omega$ is denoted by $GB(A)$ where GB stands for the greatest braid. This defines a map GB from the set of half-permutations to Ω. In particular, for any $x \in S_n$ we have $GB(L(x)) = r(x)$.

3.5. Actions of B_n of 2^{Ref} and on half-permutations. Let $R = \mathbf{R}[t^{\pm 1}]$, $q \in \mathbf{R}$, and $0 < q < 1$. The action of B_n on V defined in Sect. 3.1 has the following property: for any positive braid $x \in B_n^+$ the entries of the matrix of the map $v \mapsto xv : V \to V$ belong to $\mathbf{R}_{\geq 0} + t\mathbf{R}[t]$. (This is obvious for the generators $\sigma_1, ..., \sigma_{n-1}$ of B_n^+). Therefore the action of B_n^+ preserves the set

$$W = \bigoplus_{s \in \mathrm{Ref}} (\mathbf{R}_{\geq 0} + t\mathbf{R}[t])\, x_s \subset V.$$

For a set $A \subset \mathrm{Ref}$, define W_A as the subset of W consisting of vectors $\sum_{s \in \mathrm{Ref}} k_s x_s$ with $k_s \in \mathbf{R}_{\geq 0} + t\mathbf{R}[t]$ such that $k_s \in t\mathbf{R}[t] \Leftrightarrow s \in A$. Clearly, W is the disjoint union of the sets W_A corresponding to various $A \subset \mathrm{Ref}$. For any $x \in B_n^+$ and $A \subset \mathrm{Ref}$ there is a unique $B \subset \mathrm{Ref}$ such that $xW_A \subset W_B$. We denote this set B by xA. This defines an action of B_n^+ on 2^{Ref}. By [Kr2, Lemma 4.2], this action maps half-permutations to half-permutations. This defines an action of B_n^+ on the set of half-permutations. Finally, Krammer observes that the map GB from this set to Ω is B_n^+-equivariant. Thus, for any positive braid $x \in B_n^+$ and a half-permutation $A \subset \mathrm{Ref}$, we have

$$GB(xA) = LF(x\, GB(A)). \tag{3.2}$$

The rest of the argument is contained in the following two lemmas.

3.6. Lemma. – *Let B_n act on a set U. Suppose we are given non-empty disjoint sets $\{C_x \subset U\}_{x \in \Omega}$ such that $xC_y \subset C_{LF(xy)}$ for all $x, y \in \Omega$. Then the action of B_n on U is faithful.*

Proof. – Denote by ℓ the group homomorphism $B_n \to \mathbf{Z}$ mapping $\sigma_1, ..., \sigma_{n-1}$ to 1. We check first that the inclusion $xC_y \subset C_{LF(xy)}$ holds for all $x \in B_n^+$ and $y \in \Omega$. Clearly, $\ell(B_n^+) \geq 0$ so that we can use induction on $\ell(x)$. If $\ell(x) = 0$ then $x = 1$ and the inclusion follows from the equality $LF(y) = y$. Let $\ell(x) \geq 1$. Then $x = \sigma_i v$ where $i = 1, ..., n-1$ and $v \in B_n^+$. Clearly, $\ell(v) = \ell(x) - 1$. We have

$$xC_y = \sigma_i v C_y \subset \sigma_i(C_{LF(vy)}) \subset C_{LF(\sigma_i LF(vy))} = C_{LF(\sigma_i vy)} = C_{LF(xy)}.$$

Here the first inclusion follows from the induction hypothesis, the second inclusion follows from the assumptions of the lemma, the middle equality follows from (3.1).

It is known that for any $b \in B_n$ there are $x, y \in B_n^+$ such that $b = xy^{-1}$. Therefore to prove the lemma it suffices to show that, if two elements $x, y \in B_n^+$ act in the same way on U, then $x = y$. We will show this by induction on $\ell(x) + \ell(y)$. If $\ell(x) + \ell(y) = 0$, then $x = y = 1$. Assume that $\ell(x) + \ell(y) > 0$. By assumption, C_1 is non-empty; choose any $u \in C_1$. We have $xu \in xC_1 \subset C_{LF(x)}$ and similarly $yu \in C_{LF(y)}$. Hence $xu = yu \in C_{LF(x)} \cap C_{LF(y)}$. By the disjointness assumption, this is possible only if $LF(x) = LF(y)$. Write $z = LF(x) \in \Omega$ and consider $x', y' \in B_n^+$ such that $x = zx'$ and $y = zy'$. We have $z \neq 1$ since otherwise $x = y = 1$. Then $\ell(z) > 0$ and $\ell(x') + \ell(y') < \ell(x) + \ell(y)$. The induction assumption yields that $x' = y'$. Therefore $x = y$. This proves the inductive step and the lemma.

3.7. Lemma. – *For $x \in \Omega$, set*

$$C_x = \bigcup_{A \in GB^{-1}(x)} W_A \subset V$$

where A runs over all half-permutations such that $GB(A) = x$. Then the sets $\{C_x\}_{x \in \Omega}$ satisfy all the conditions of Lemma 3.6. Therefore the action of B_n on V is faithful.

Proof. – It is obvious that the sets $\{C_x\}_{x \in \Omega}$ are disjoint. The set C_x is non-empty because $\emptyset \neq W_{L(r^{-1}(x))} \subset C_x$.

To prove the inclusion $xC_y \subset C_{LF(xy)}$ for $x, y \in \Omega$, it suffices to prove that $xW_A \subset C_{LF(xy)}$ whenever A is a half-permutation such that $GB(A) = y$. This follows from the inclusions

$$xW_A \subset W_{xA} \subset C_{GB(xA)} = C_{LF(xy)}.$$

Here the first inclusion follows from the definition of the B_n^+-action of Ref. The second inclusion follows from the definition of $C_{GB(xA)}$. The equality follows from (3.2).

3.8. More about the representation. – Let $\ell_\Omega : B_n \to \mathbf{Z}$ be the length function with respect to the generating set Ω: for $x \in B_n$, $\ell_\Omega(x)$ is the minimal natural number k such that $x = x_1 ... x_k$ where $x_i \in \Omega$ or $x_i^{-1} \in \Omega$ for each $i = 1, ..., k$. Among other related results, Krammer gives an explicit computation of ℓ_Ω in terms of his representation. Namely, take the Laurent polynomial ring $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ as the ground ring and denote by ρ Krammer's representation $B_n \to \mathrm{Aut}(V)$ defined in Sect. 3.1. For $x \in B_n$, consider the Laurent expansion $\rho(x) = A_k t^k + A_{k+1} t^{k+1} + ... + A_l t^l$ where $\{A_i\}_{i=k}^l$ are $(m \times m)$-matrices over $\mathbf{Z}[q^{\pm 1}]$ and $A_k \neq 0, A_l \neq 0$. Then

$$\ell_\Omega(x) = \max(l - k, l, -k).$$

This formula yields another proof of the faithfulness of ρ: If $x \in \mathrm{Ker}\rho$, then $k = l = 0$ so that $\ell_\Omega(x) = 0$ and $x = 1$.

The length function ℓ_Ω was first considered by Charney [Ch] who proved that the formal power series $\sum_{x \in B_n} z^{\ell_\Omega(x)} \in \mathbf{Z}[[z]]$ is a rational function. It is unknown whether the similar formal power series determined by the length function with respect to the generators $\sigma_1, ..., \sigma_{n-1}$ is rational.

4. BMW-ALGEBRAS AND REPRESENTATIONS OF B_n

4.1. Hecke algebras and representations of B_n. A vast family of representations of B_n including the Burau representation arise from a study of Hecke algebras. The Hecke algebra $H_n(\alpha)$ corresponding to $\alpha \in \mathbf{C}$ can be defined as the quotient of the complex group ring $\mathbf{C}[B_n]$ by the relations $\sigma_i^2 = (1-\alpha)\sigma_i + \alpha$ for $i = 1, ..., n-1$. This family of finite dimensional $\mathbf{C}$-algebras is a one-parameter deformation of $\mathbf{C}[S_n] = H_n(1)$. For α sufficiently close to 1, the algebra $H_n(\alpha)$ is isomorphic to $\mathbf{C}[S_n]$. For such α, the algebra $H_n(\alpha)$ is semisimple and its irreducible representations are indexed by the Young diagrams with n boxes. Their decomposition rules and dimensions are the same as for the irreducible representations of S_n.

Each representation of $H_n(\alpha)$ yields a representation of B_n via the natural projection $B_n \subset \mathbf{C}[B_n] \to H_n(\alpha)$. This gives a family of irreducible finite dimensional representations of B_n indexed by the Young diagrams with n boxes. These representations were extensively studied by V. Jones [Jo]. In particular, he observed that the $(n-1)$-dimensional Burau representation of B_n appears as the irreducible representation associated with the two-column Young diagram whose columns have $n-1$ boxes and one box, respectively.

4.2. Birman-Murakami-Wenzl algebras and their representations. Jun Murakami [Mu] and independently J. Birman and H. Wenzl [BW] introduced a two-parameter family of finite dimensional $\mathbf{C}$-algebras $C_n(\alpha, l)$ where α and l are non-zero complex numbers such that $\alpha^4 \neq 1$ and $l^4 \neq 1$. For $i = 1, ..., n-1$, set

$$e_i = (\alpha + \alpha^{-1})^{-1}(\sigma_i + \sigma_i^{-1}) - 1 \in \mathbf{C}[B_n].$$

The algebra $C_n(\alpha, l)$ is the quotient of $\mathbf{C}[B_n]$ by the relations

$$e_i\sigma_i = l^{-1}e_i, \quad e_i\sigma_{i-1}^{\pm 1}e_i = l^{\pm 1}e_i$$

for all i. (The original definition in [BW] involves more relations; for the shorter list given above, see [We]). The algebra $C_n(\alpha, l)$ admits a geometric interpretation in terms of so-called Kauffman skein classes of tangles in Euclidean 3-space. This family of algebras is a deformation of an algebra introduced by R. Brauer [Br] in 1937.

The algebraic structure and representations of $C_n(\alpha, l)$ were studied by Wenzl [We], who established (among other results) the following three facts.

(i) *For generic α, l, the algebra $C_n(\alpha, l)$ is semisimple.*

Here "generic" means that α is not a root of unity and $\sqrt{-1}\,l$ is not an integer power of $-\sqrt{-1}\,\alpha$. (The latter two numbers correspond to r and q in Wenzl's notation). In the sequel we assume that α, l are generic in this sense. We denote the number of boxes in a Young diagram λ by $|\lambda|$.

(ii) *The irreducible finite dimensional $C_n(\alpha, l)$-modules are indexed by the Young diagrams λ such that $|\lambda| \leq n$ and $|\lambda| \equiv n \pmod 2$.*

The irreducible $C_n(\alpha, l)$-module corresponding to λ will be denoted by $V_{n,\lambda}$. Composing the natural projection $B_n \subset \mathbf{C}[B_n] \to C_n(\alpha, l)$ with the action of $C_n(\alpha, l)$ on $V_{n,\lambda}$ we obtain an irreducible representation of B_n.

Observe that the inclusion $B_{n-1} \hookrightarrow B_n$ sending each $\sigma_i \in B_{n-1}$ with $i = 1, ..., n-2$ to $\sigma_i \in B_n$ induces an inclusion $C_{n-1}(\alpha, l) \hookrightarrow C_n(\alpha, l)$ for all $n \geq 2$.

(iii) *The $C_n(\alpha, l)$-module $V_{n,\lambda}$ decomposes as a $C_{n-1}(\alpha, l)$-module into a direct sum $\oplus_\mu V_{n-1,\mu}$ where μ ranges over all Young diagrams obtained by removing or (if $|\lambda| < n$) adding one box to λ. Each such μ appears in this decomposition with multiplicity 1.*

4.3. The Bratelli diagram for the BMW-algebras. The assertions (ii) and (iii) in Sect. 4.2 allow us to draw the Bratelli diagram for the sequence $C_1(\alpha, l) \subset C_2(\alpha, l) \subset ...$ On the level $n = 1, 2, ...$ of the Bratelli diagram one puts all Young diagrams λ such that $|\lambda| \leq n$ and $|\lambda| \equiv n \,(\mathrm{mod}\, 2)$. Then one connects by an edge each λ on the n-th level to all Young diagrams on the $(n-1)$-th level obtained by removing or (if $|\lambda| < n$) adding one box to λ. For instance, the $n = 1$ level consists of a single Young diagram with one box corresponding to the tautological one-dimensional representation of $C_1(\alpha, l) = \mathbf{C}$. The $n = 2$ level contains the empty Young diagram and two Young diagrams with two boxes. All three are connected by an edge to the diagram on the level 1. Note that every Young diagram λ appears on the levels $|\lambda|, |\lambda| + 2, |\lambda| + 4, ...$

The Bratelli diagram yields a useful method of computing the dimension of $V_{n,\lambda}$ where λ is a Young diagram on the n-th level. It is clear from (iii) that $\dim(V_{n,\lambda})$ is the number of paths on the Bratelli diagram leading from λ to the only diagram on the level 1. Here by a path we mean a path with vertices lying on consecutively decreasing levels. We give three examples of computations based on (iii).

(a) Let λ_n be the Young diagram represented by a column of n boxes. There is only one path from λ_n, positioned on the level n, to the top of the Bratelli diagram. Hence, $\dim(V_{n,\lambda_n}) = 1$ for all $n \geq 1$.

For $n \geq 2$, the algebra $C_n(\alpha, l)$ has two one-dimensional representations. In both of them all e_i act as 0 and all σ_i act as multiplication by one and the same number equal either to α or to α^{-1}. We choose the correspondence between the irreducible $C_n(\alpha, l)$-modules and the Young diagrams so that all σ_i act on V_{n,λ_n} as multiplication by α. If λ_n^T is the Young diagram represented by a row of n boxes, then similarly to (a) we have that $\dim(V_{n,\lambda_n^T}) = 1$ and all σ_i act on V_{n,λ_n^T} as multiplication by α^{-1}.

(b) For $n \geq 2$, let λ'_n be the two-column Young diagram whose columns have $n-1$ boxes and one box, respectively. For $n \geq 3$, the diagram λ'_n, positioned on the level n, is connected to only two Young diagrams on the previous level, namely, to λ'_{n-1} and λ_{n-1}. Hence

$$\dim(V_{n,\lambda'_n}) = \dim(V_{n-1,\lambda'_{n-1}}) + \dim(V_{n-1,\lambda_{n-1}}) = \dim(V_{n-1,\lambda'_{n-1}}) + 1.$$

We have $\lambda'_2 = \lambda_2^T$ so that $\dim(V_{2,\lambda'_2}) = 1$. Hence $\dim(V_{n,\lambda'_n}) = n - 1$ for all $n \geq 2$.

(c) For $n \geq 2$, consider the module $V_{n,\lambda_{n-2}}$ corresponding to the Young diagram λ_{n-2} positioned on the level n. If $n \geq 3$, then this diagram is connected to three Young diagrams on the previous level, namely, to $\lambda_{n-1}, \lambda_{n-3}, \lambda'_{n-1}$. Hence

$$\dim(V_{n,\lambda_{n-2}}) = \dim(V_{n-1,\lambda_{n-1}}) + \dim(V_{n-1,\lambda_{n-3}}) + \dim(V_{n-1,\lambda'_{n-1}})$$

$$= \dim(V_{n-1,\lambda_{n-3}}) + n - 1.$$

We gave $\lambda_0 = \emptyset$ and by (iii) above, $\dim(V_{2,\lambda_0}) = \dim(V_{1,\lambda_1}) = 1$. Thus for all $n \geq 2$,

$$\dim(V_{n,\lambda_{n-2}}) = n(n-1)/2,$$

i.e., $V_{n,\lambda_{n-2}}$ has the same dimension as the Krammer representation of B_n. We now rescale the representation $B_n \to \mathrm{Aut}(V_{n,\lambda_{n-2}})$ by dividing the action of each σ_i by α.

4.4. Theorem. (M. Zinno [Zi]) – *The Krammer representation corresponding to* $q = -\alpha^{-2}$ *and* $t = \alpha^3 l^{-1}$ *is isomorphic to the rescaled representation* $B_n \to \mathrm{Aut}(V_{n,\lambda_{n-2}})$.

The proof given in [Zi] goes by a direct comparison of both actions of B_n on certain bases. Theorem 4.4 implies that the Krammer-Bigelow representation considered in Sect. 2 and 3 is irreducible.

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